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# INTERPOLATION AND APPROXIMATION FROM THE $M$-THEORY POINT OF VIEW

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The purpose of the present paper is to extend some basic results from function algebras theory to the context of $M$-theory. Particularly we are able to prove that Bishop-Silov decomposition still works in full capacity. That offers a satisfactory explanation for the similarities between approximation results in function algebras theory, $C^*$-algebra theory and convexity theory.
The basic idea behind our technical construction is as follows : If $K$ is a compact Hausdorff space then the closed ideals of $C(K, \mathbb{C})$ are in a one-to-one correspondence to the annihilators of closed subsets of $K$ and moreover, if $I_H = \{f \mid f|H = 0\}$ is such an annihilator, then $C(K, \mathbb{C})/I_H$ can be identified with $C(H, \mathbb{C})$ and the canonical quotient mapping $C(K, \mathbb{C}) \to C(K, \mathbb{C})/I_H$ is nothing but the restriction mapping $f \to f|H$. That fact is strongly related to Urysohn Lemma. Actually most of our work can be viewed as a contribution to non-commutative topology.
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## 1. PRELIMINARIES ON FUNCTION ALGEBRAS

We survey in this section some basic facts on interpolation and approximation in function algebra theory. The details will be found in Gamelin [9] or Suciu [13].

Let $K$ be a compact Hausdorff space. By a *function algebra* on $K$ we mean any closed subalgebra $\mathscr{A}$ of $C(K, \mathbb{C})$ that contains the constants and separates the points of $K$. An important example (beside $C(K, \mathbb{C})$) is $A(D)$, the Banach algebra of all analytic functions on the unit disc $D$, that admit a continuous extension to $\bar{D}$.

The interpolation theory aims to outline the closed subsets $H$ of $K$ on which the action of a given function algebra $\mathscr{A}$ can be well controlled. To be more specific, put

$$H^{\perp} = \{f \mid f \in \mathscr{A},\ f \mid H = 0\};$$

and

$$\mathscr{A} \mid H = \{f \mid H \mid f \in \mathscr{A}\},$$

viewed as a subspace of $C(H, \mathbb{C})$. The natural mapping $T : \mathscr{A}/H^{\perp} \to \mathscr{A} \mid H$, $T(\hat{f}) = f|H$ for $f \in \mathscr{A}$, is an algebraic isomorphism whose norm is $\leqslant 1$. We shall say that $H$ is a *set of strict interpolation* with respect to $\mathscr{A}$ provided that $T$ is an isometry. e.g., this is the case if $H$ is an intersection of peak sets. Recall that a closed subset $H$ of $K$ is a *peak set* (with respect to $\mathscr{A}$) provided that there exists a function $f \in \mathscr{A}$ such that $f|H = 1$ and $|f(x)| < 1$ for $x \notin H$.

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The definition of a strict interpolation set makes sense for any closed subspace $\mathscr{A}$ of $C(K, \mathbb{C})$. In this context the role of intersections of peak subsets is played by frontal subsets. A closed subset $H$ of $K$ is called a *frontal set* (with respect to $\mathscr{A}$) provided that for every $f \in \mathscr{A}$, every neighbourhood $V$ of $H$ and every couple $(\varepsilon, \delta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, there exists a function $\tilde{f} \in \mathscr{A}$ such that

$$\tilde{f} \mid H = f$$

$$\sup_K |\tilde{f}(x)| \leqslant \sup_H |f(x)| + \delta$$

$$\sup_{K \setminus V} |\tilde{f}(x)| \leqslant \varepsilon.$$

In section 3 we shall discuss a generalization of this concept based on the following result:

**1.1** LEMMA (See [4], p. 364). *A closed subset $H$ of a compact Hausdorff space $K$ is frontal (with respect to a closed subspace $\mathscr{A}$ of $C(K, \mathbb{C})$) if and only if the mapping $\mu \to \chi_H \cdot \mu$ from $C(K, \mathbb{C})'$ into $C(K, \mathbb{C})'$ leaves invariant $\mathscr{A}^\circ$*, the polar of $\mathscr{A}$.

A subset $H$ of $K$ is called a *set of anti-symmetry* of the function algebra $\mathscr{A}$ provided that $f \in \mathscr{A}$ and $\bar{f} \mid H$ is a real function implies $f \mid H$ is constant. The function algebra $\mathscr{A}$ is called anti-symmetric provided that $K$ itself is a set of anti-symmetry.

The closure of a set of anti-symmetry is also a set of anti-symmetry. Every maximal set of anti-symmetry is closed. Every point $x \in K$ belongs to a maximal set of anti-symmetry.

**1.2** DE BRANGES' LEMMA. *Let $\mathscr{A}$ be a function algebra on $K$ and let $\mu$ be an extreme point of the unit ball of $\mathscr{A}^\circ$. Then* Supp $\mu$, *the support of $\mu$, is a set of anti-symmetry of $\mathscr{A}$.*

**1.3** BISHOP-SILOV DECOMPOSITION. *Let $\mathscr{A}$ be a function algebra on $K$. Then $K$ admits a decomposition $K = \bigcup K_\alpha$ where $(K_\alpha)_\alpha$ is the set of all maximal subsets of anti-symmetry of $\mathscr{A}$. Then $K_\alpha \cap K_\beta = \emptyset$ for $\alpha \neq \beta$ and moreover.*

(a) *$\mathscr{A} \mid K_\alpha$ is a closed subspace of $C(K_\alpha, \mathbb{C})$ for every $\alpha$;*

(b) *$f \in C(K, \mathbb{C})$ belongs to $\mathscr{A}$ if and only if $f \mid K_\alpha \in \mathscr{A} \mid K_\alpha$ for every $\alpha$.*

In 1978, Păltineanu [11] extended the Bishop-Silov decomposition by considering instead of function algebras $\mathscr{A}$ on $K$, closed subspaces of $C(K, \mathbb{C})$. In section 4 we shall prove that Bishop-Silov decomposition still works in the framework of $M$-structure theory.

## 2. REVIEW ON $M$-STRUCTURE THEORY

Roughly speaking, $M$-structure theory measures to what extent a given Banach space behaves like a space of continuous on a compact Hausdorff space (i.e. a Kakutani $M$-space with unit). In what follows we review some basic facts which we shall need in our approach to abstract interpolation. We refer to [3] for a thorough presentation of $M$-structure theory. See also [1], [2].

Let $E$ be a Banach space. An idempotent $P \in L(E, E)$ is called an *L-projection* provided that

$$\|x\| = \|Px\| + \|x - Px\| \quad \text{for all } x \in E.$$

The $L$-projections on $E$ commute and thus they constitute a Boolean algebra of projections on $E$ (denoted by $\mathbb{P}_L(E)$) by letting

$$P \vee Q = P + Q - PQ$$

$$P \wedge Q = PQ$$

$$P^{\perp} = I - P.$$

Actually $\mathbb{P}_L(E)$ is *Bade complete* i.e., for every familly $(P_\alpha)_\alpha$ of elements of $\mathbb{P}_L(E)$ there exist $\vee P_\alpha$ and $\wedge P_\alpha$ in $\mathbb{P}_L(E)$ and moreover

$$(\vee P_\alpha)(E) = \overline{\text{Span}} \cup P_\alpha(E)$$

$$(\wedge P_\alpha)(E) = \cap P_\alpha(E).$$

A closed subspace $I$ of $E$ is said to be an *M-ideal* provided that its polar $I^\circ$ is the image of an *L-projection on* $E'$. $I^\circ$ can be the image of at most one $L$-projection (usually denoted by $P_I$). This is a consequence of the following

2.1 Lemma *Let $P$ and $Q$ be two projections of $\mathbb{P}_L(E')$ such that* $\text{Im } P = \text{Im } Q$. *Then* $P = Q$.

*Proof*. In fact, for every $x \in E$ there exists a $y \in E$ such that $Px = Qy$. Then $(I - Q)Px = (I - Q)Qy = 0$ and in the same manner we can prove that $(I - P)Q = 0$. Since $PQ = QP$, it follows that $P = QP = PQ = Q$. ■

If $E$ is a $C^*$-algebra then its $M$-ideals are precisely the closed two-sided ideals. If $E$ is an $M$-space then its $M$-ideals coincide with the closed lattice ideals. See [2].

If $E$ is the space $A(K, \mathbb{R})$ (of all continuous affine real functions on a compact convex subset of a locally convex Hausdorff space) then the $M$-ideals of $E$ are the annihilators of the split faces of K. See [2].

If $E$ is a Lindenstrauss space (i.e., if $E'$ is an $L^1$-space), then the $M$-ideals of $E$ are the annihilators of the bifaces of the closed unit ball of $E'$. See [2].

If $E$ is a function algebra on a compact metrizable space $K$, then the $M$-ideals of $E$ are exactly the annihilators of the peak sets. See [10].

We shall denote by $M(E)$ the set of all $M$-ideals of a Banach space $E$. The map $I \to P_I$, from $M(E)$ onto $\mathbb{P}_L(E')$ is bijective and thus $M(E)$ can organized naturally as a Boolean algebra.

2.2 Lemma. (i) *Every finite sum as well as every finite intersection of $M$-ideals of $E$ is still an $M$-ideal of $E$.*

(ii) *If* $(I_\alpha)_\alpha$ *is a family of elements of* $M(E)$ *then* $\overline{\text{Span}} \cup I_\alpha \in M(E)$.

By Lemma 2.2. ii, every closed subspace of $E$ contains a largest $M$-ideal.

In contrast to the situation for ideals in rings, arbitrary intersections of $M$-ideals need not be an $M$-ideal. See [5]. We call a Banach space $E$ *$M$-distinguished* provided that $M(E)$ is closed under arbitrary intersections. Examples are $C^*$-algebras, $M$-spaces, $G$-spaces (see [14]), reflexive Banach spaces, etc.

**2.3** Lemma. *Let* $I \in M(E)$. *Then:*

(i) *The $M$-ideals of $I$ are precisely the $M$-ideals of $E$ that are contained in $I$.*

(ii) *Let* $\pi_I : E \to E/I$ *be the canonical mapping. Then* $\pi_I^{-1}$ *maps the $M$-ideals of $E/I$ into $M$-ideals of $E$.*

(iii) *The $M$-ideals of $E/I$ are just the canonical images of the $M$-ideals of $E$.*

To any Banach space $E$ one can associate two operator algebras. The first one is the so-called *Cunningham algebra*,

$$C(E) = \overline{\text{Span}}\, \mathbb{P}_L(E),$$

the closure being taken in the uniform topology of $L(E, E)$. $C(E)$ is a commutative Banach algebra with unit $1_E$, the identify of $E$. Also, $C(E)$ is algebraic and isometric isomorphic to $C(\text{Spec}\, \mathscr{B}, K)$, where Spec $\mathscr{B}$ denotes the Stone space associated to $\mathscr{B} = \mathbb{P}_L(E)$ and $K$ denotes the field of scalars. See Evans [8] for details. Particularly, $C(E)$ is a Banach lattice, possibly complex. If we denote by Re $C(E)$ the closure of the finite real combinations of elements of $\mathbb{P}_L(E)$ the above isomorphism induces an isomorphism

$$\text{Re}\ C(E) \rightleftarrows C(\text{Spec}\, \mathbb{P}_L(E), \mathbb{R}).$$

The second algebra is *the centralizer*. It is the Banach subalgebra $Z(E)$ of $L(E, E)$ consisting of all operators $T \in L(E, E)$ such that $T' \in C(E')$. $Z(E)$ is also a commutative Banach algebra with unit $1_E$ and each $T \in Z(E)$ leaves invariant every $M$-ideal of $E$.

We define the real part of the centralizer by

$$\text{Re}\ Z(E) = \{T \mid T' \in \text{Re}\ C(E')\}.$$

It is natural to consider on Re $Z(E)$ the order relation

$$S \leqslant T \text{ in } \text{Re}\, Z(E) \text{ if and only if } S' \leqslant T' \text{ in}$$

$$\text{Re}\ C(E') = C(\text{Spec}\, \mathbb{P}_L(E'), \mathbb{R}).$$

With respect to this order relation Re $Z(E)$ becomes a $C(S, \mathbb{R})$-space. Alfsen and Effros [2] have described the order relations on Re $C(E)$ and Re $Z(E)$ via order relations on $E$. We shall not enter the details here. However it seems worthwhile to recall their basic remark.

Consider on the Banach space $E$ the *L-order relation*,

$$x \ll_L y \text{ if and only if } \|y\| = \|x\| + \|y - x\|.$$

Then $0 \leqslant S \leqslant T$ in $\operatorname{Re} C(E)$ if and only if $Sx \ll_L Tx$ for every $x \in E$. A consequence of this remark is the following

2.4 PROPOSITION. *Let $T \in \operatorname{Re} Z(E)$ such that $0 \leqslant T \leqslant I$. Then* $\operatorname{Im} T$ *is an $M$-ideal.*

*Proof.* By hypothesis, $0 \leqslant T' \leqslant I$ in $\operatorname{Re} C(E')$. We have to show that $(\operatorname{Im} T)^\circ = \operatorname{Ker} T'$ is an $L$-summand. Since $\operatorname{Re} C(E') = C(\operatorname{Spec}\mathbb{P}_L(E'), \mathbb{R})$, there exists an increasing sequence $(S_n)_n$ of finite linear combinations of $\mathbb{P}_L(E')$ with positive coefficients such that $\|S_n - T'\| \to 0$. Then $S_n x \ll_L T'x$ for every $x \in E$ and thus $\operatorname{Ker} T' \subset \cap \operatorname{Ker} S_n$. Actually the equality holds because $\|S_n - T'\| \to 0$.

Each $\operatorname{Ker} S_n$ is a finite intersection of $L$-summands and every intersection of $L$-summands is still an $L$-summand, which ends the proof. ■

In what follows we shall discuss the case of $C^*$-algebras $\mathcal{U}$ with unit $\mathbb{1}$. In this case $Z(\mathcal{U})$ coincides with the center of $\mathcal{U}$,

$$Z(\mathcal{U}) = \{x \mid x \in \mathcal{U} \text{ and } xy = yx \text{ for every } y \in \mathcal{U}\}.$$

and

$$\operatorname{Re} Z(\mathcal{U}) = \{x \mid x \in \mathcal{U},\ -\alpha\cdot\mathbb{1} \leqslant x \leqslant \alpha\cdot\mathbb{1} \text{ for a suitable } \alpha \geqslant 0\}.$$

Moreover, the operatorial norm on $\operatorname{Re} Z(\mathcal{U})$ coincides with the norm

$$\|x\| = \inf\{\alpha \mid -\alpha\cdot\mathbb{1} \leqslant x \leqslant \alpha\cdot\mathbb{1}\}.$$

$\operatorname{Re} Z(\mathcal{U})$ constitutes a Banach lattice with strong order unit when endowed with the order induced by $\operatorname{Re} \mathcal{U}$. Consequently Kakutani's representation theorem applies to $\operatorname{Re} Z(\mathcal{U})$ and thus $\operatorname{Re} Z(\mathcal{U})$ can be viewed as a $C(S, \mathbb{R})$-space. The details of all above assertions are to be found in Wils [15].

## 3. FRONTAL IDEALS

Our generalization of the notion of a frontal ideal is motivated by Lemma 1.1 above.

In the sequel $E$ will denote a Banach space, $X$ and $I$ closed subspaces of $E$ and $\pi_I : E \to E/I$ the canonical quotient mapping.

**3.1** DEFINITION. An $M$-ideal $I$ of $E$ is said to be an ($X$-) *frontal ideal* provided that the $L$-projection $P_I \in L(E', E')$, whose image is $I^\circ$, leaves invariant $X^\circ$.

By Lemma 1.1 above, a closed subset $H$ of a compact Hausdorff space $K$ is frontal (with respect to a closed subspace $X$ of $C(K, \mathbb{C})$) if and only if $I_H$, the annihilator of $H$, is an $X$-frontal ideal of $C(K, \mathbb{C})$.

We shall denote by $F_X(E)$ the set of all $X$-frontal ideals of $E$. $F_E(E)$ is simply $M(E)$. If $I$ and $J$ are in $M(E)$, then $I \in F_J(E)$; in fact, any two $L$-projections commute. If $I \in M(E)$ and $X$ is a closed subspace of $E$ such that $X \subset I$ or $I \subset X$ then $I \in F_X(E)$.

$F_X(E)$ is closed under finite sums and finite intersections. See Lemma 2.2 above.

Frontal ideals satisfy a weaker analogue of Tietze-Urysohn Extension Theorem.

**3.2** Proposition. *Suppose that $I$ is an $X$-frontal ideal of $E$ and $J$ is an $M$-ideal of $E$ such that $E = I + J$. Then for each $x \in X$ and each $\varepsilon > 0$ there exists an $\bar{x} \in X$ such that*

$$\pi_I(\bar{x}) = \pi_I(x)$$

$$\|\bar{x}\| \leqslant \|\pi_I(x)\| + \varepsilon$$

$$\|\pi_J(\bar{x})\| \leqslant \varepsilon.$$

*Proof.* Notice first that $X$ can be renormed by

$$|||x||| = \max\left\{\|x\|, \frac{1}{\varepsilon}\|\pi_J(x)\|\right\}.$$

Let $X_1$ be the closed unit ball of $(X, |||\ |||)$. Our next step is to prove the following assertion :

(*) $\pi_I(X_1)$ is dense in $\{\pi_I(x) \mid x \in X,\ \|\pi_I(x)\| \leqslant 1\}$.

For, let $f$ $\pi_I(X_1)^\circ$. Then

$$|f(\pi_I(x))| \leqslant \max\left\{\|x\|, \frac{1}{\varepsilon}\|\pi_J(x)\|\right\} \quad \text{for } x \in X$$

and an easy consequence of Hahn-Banach extension theorem shows that $g = f \circ \pi_I | X$ admits a decomposition $g = g_1 + g_2$ where $|g_1(x)| \leqslant \|x\|$ and $|g_2(x)| \leqslant \varepsilon^{-1} \cdot \|\pi_J(x)\|$ for all $x \in X$.

Let $f_1$ and $f_2$ be linear extensions of $g_1$ and $g_2$ respectively to $E$ such that $\|f_1\| = \|g_1\|$ and $\|f_2\| = \|g_2\|$. Then

(i) $$f \circ \pi_I - f_1 - f_2 \in X^\circ$$

and

(ii) $$f_2 \in J^\circ.$$

Since $I + J = E$, from (ii) we infer that $P_I(f_2) = P_I P_J(f_2) = 0$. Since $f \circ \pi_I \in I^\circ$ and $P_I(X^\circ) \subset X^\circ$, we infer from (i) and (ii) that $f \circ \pi_I - P_I(f_1) \in X^\circ$. Then

$$f \mid \pi_I(X) = P_I(f_1) \mid \pi_I(X)$$

so that for every $x \in X$ with $\|\pi_I(x)\| \leqslant 1$ we have

$$|f(\pi_I(x))| = |P_I(f_1)(\pi_I(x)) \leqslant \| P_I(f_1)\| \leqslant \|f_1\| \leqslant 1.$$

Consequently $\pi_I(X_1)$ is dense in $\{\pi_I(x) \mid x \in X, \ \|\pi_I(x)\| \leqslant 1\}$ and assertion $(*)$ is proved.

To end the proof of Proposition 3.2, let $x \in X$ and let $\varepsilon > 0$. By $(*)$ there exists a $y \in X$ such that $\|y\| \leqslant 1$, $\|\pi_J(y)\| \leqslant \varepsilon/2\alpha$ and $\|\pi_I(x/\alpha - y)\| < \varepsilon/2\alpha$, where $\alpha = \|\pi_1(x)\|$. Choose $u \in I$ such that

$$\| x - \alpha y - \alpha u \| < \varepsilon/2$$

Then $\bar{x} = x - \alpha u$ verifies the equality $\pi_I(\bar{x}) = \pi_I(x)$ and moreover

$$\|\bar{x}\| \leqslant \alpha \|y\| + \|x - \alpha u - \alpha y\| \leqslant \|\pi_I(x)\| + \varepsilon$$

and

$$\|\pi_J(\bar{x})\| \leqslant \|\pi_J(\alpha y)\| + \|x - \alpha u - \alpha y\| \leqslant \varepsilon. \quad \blacksquare$$

**3.3** COROLLARY. *Let $I$ and $J$ be two $M$-ideals of $E$ such that $E = I + J$. Then for each $x \in E$ and each $\varepsilon > 0$ there exists an $\bar{x} \in E$ such that*

$$\pi_I(\bar{x}) = \pi_I(x)$$

$$\|\bar{x}\| \leqslant \|\pi_I(x)\| + \varepsilon$$

$$\|\pi_J(\bar{x})\| \leqslant \varepsilon.$$

In the sequel we shall make use of the fact that the restriction to $X$ of the canonical quotient mapping $\pi_I : E \to E/I$ admits a natural factorization

$$X \xrightarrow{L_I} X/X \cap I \xrightarrow{R_I} \pi_I(X).$$

The mappings $L_I$ and $R_I$ are both continuous when $X/X \cap I$ is endowed with the quotient norm.

**3.4** *Definition.* An $M$-ideal $I$ of $E$ is said to be a *strict interpolating subspace* for $X$ provided that the mapping $R_I : X/X \cap I \to \pi_I(X)$ mentionned above is an algebraic isometric isomorphism.

Since $X/X \cap I$ is complete, it follows that $\pi_I(X)$ is a closed subspace of $E/I$ whenever $I$ is a strict interpolating subspace for $X$. Compare to Theorem 1.3 a.

Quite obvious, $I$ is a strict interpolating subspace for $X$ if and only if

$$\pi_I(B \cap X) = \pi_I(B) \cap \pi_I(X),$$

where $B$ denotes the open unit ball of $E$.

We shall show that every $X$-frontal ideal is also a strict interpolating ideal for $X$. We need to restate Definition 3.4 in a more convenient form.

**3.5 Lemma.** *Let $E$ be a Banach space and let $B$ be its open unit ball. An $M$-ideal $I$ of $E$ is an interpolation ideal for $X$ if and only if $\pi_I(B \cap X)$ is dense in $\pi_I(B) \cap \pi_I(X)$.*

*Proof.* The necessity is clear.

The sufficiency. We shall prove first that

$$\pi_I(3^{-1}B) \cap \pi_I(X) \subset \pi_I(B \cap X). \tag{**}$$

Let $y \in \pi_I(3^{-1}B) \cap \pi_I(X)$. Since $\pi_I(3^{-1}B \cap X)$ is dense in $\pi_I(3^{-1}B) \cap \pi_I(X)$, there exists an $x_1 \in 3^{-1}B \cap X$ such that $y - \pi_I(x_1) \in \pi_I(3^{-2}B)$. Since $\pi_I(3^{-2}B \cap X)$ is dense in $\pi_I(3^{-2}B) \cap \pi_I(X)$, there exists an $x_2 \in 3^{-2}B \cap X$ such that $y - \pi_I(x_1) - \pi_I(x_2) \in \pi_I(3^{-3}B)$, and so on. Consequently there exists a sequence $(x_n)_n$ of elements of $E$ such that $x_n \in 3^{-n}B \cap X$ and

$$y - \sum_{k=1}^{n} \pi_I(x_k) \in \pi_I(3^{-n-1}B)$$

for every $n \in \mathbb{N}^*$.

Since $X/X \cap I$ is complete, the series $\sum_{n=1}^{\infty} L_I(x_n)$ is convergent to an $\bar{x} \in X/X \cap I$. $E/I$ being separated and $R_I$ continuous, it follows that

$$R_I(\bar{x}) = \sum_{n=1}^{\infty} R_I L_I(x_n) = \sum_{n=1}^{\infty} \pi_I(x_n) = y.$$

The inclusion $3\, L_I(3^{-1}B \cap X) \subset L_I(B \cap X)$ yields

$$L_I(3^{-1}B \cap X) + L_I(3^{-1}B \cap X) \subset L_I(B \cap X).$$

Then $\sum_{k=1}^{\infty} L_I(x_k) \in L_I(3^{-1}B \cap X) + L_I(3^{-1}B \cap X)$ for every $n \in \mathbb{N}^*$ so that $\bar{x} \in L_I(B \cap X)$.

Let $x \in B \cap X$ such that $L_I(x) = \bar{x}$. Then

$$y = R_I(\bar{x}) = R_I L_I(x) = \pi_I(x) \in \pi_I(B \cap X)$$

which ends the proof of $(**)$.

By $(**)$,

$$\pi_I(B) \cap \pi_I(X) \subset \overbrace{\pi_I(B \cap X)}^{\circ} \subset \pi_I(B \cap X) \subset \pi_I(B) \cap \pi_I(X)$$

i.e., $\pi_I(B \cap X) = \pi_I(B) \cap \pi_I(X)$ so that $I$ is a strict interpolating ideal for $X$. ■

**3.6** THEOREM. *Every $X$-frontal ideal $I$ of $E$ is also a strict interpolating ideal for $X$.*

*Proof.* By Lemma 3.5, it suffices to show that

$$(\pi_I(B \cap X))^\circ \subset (\pi_I(B) \cap \pi_I(X))^\circ.$$

Let $f \in (\pi_I(B \cap X))^\circ$. Then $\pi_I'(f) \in (B \cap X)^\circ$ and thus there exists a $g \in B^\circ$ such that $g - \pi_I'(f) \in X^\circ$.

Since $\pi_I'(f) \in I^\circ$, we infer that $(P_I \circ \pi_I')(f) = \pi_I'(f)$ and

$$P_I(g) - \pi_I'(f) = P_I(g - \pi_I'(f)) \in X^\circ \cap I^\circ.$$

For every $y \in \pi_I(B) \cap \pi_I(X)$ there exists $x_1 \in B$ and $x_2 \in X$ such that $y = \pi_I(x_1) = \pi_I(x_2)$. Then

$$f(y) = f(\pi_I(x_1)) = (P_I(g))(x_1) + (\pi_I'(f) - P_I(g))(x_1) =$$
$$= (P_I(g))(x_1) + (\pi_I'(f) - P_I(g))(x_2)$$

which shows that $f \in (\pi_I(B) \cap \pi_I(X))^\circ$. ■

**3.7** COROLLARY. *If $I$ is an $X$-frontal ideal of $E$, then $\pi_I(X)$ is a closed subspace of $E/I$.*

**3.8** COROLLARY. *If $I$ is an $X$-frontal ideal of $E$, then for every $x \in X$ there exists an $\bar{x} \in X$ such that $\pi_I(\bar{x}) = \pi_I(x)$ and $\|\bar{x}\| = \|\pi_I(x)\|$ (i.e., $X \cap I$ is proximinal in $X$).*

By Corollary 3.8, for every $I \in M(E)$, the canonical quotient mapping $\pi_I : E \to E/I$ maps the closed unit ball of $E$ onto the closed unit ball of $E/I$. This fact was first noticed in [2]. See [1] for the complex case.

Corollary 3.8 suggests that the result of Corollary 3.3 above might be strengthen up to $\|\bar{x}\| = \|\pi_I(x)\|$.

The following result is an analogue of the fact that frontal subsets of frontal subsets are frontal.

**3.9** PROPOSITION. *Let $I$ and $J$ be two $M$-ideals of $E$ such that $I \subset J$ and $I \in F_X(E)$. Then $J \in F_X(E)$ if and only if $J/I \in F_{\pi_I(X)}(E/I)$.*

*Proof.* By Corollary 3.7 above, $\pi_I(X)$ is a closed subspace of $E/I$. Another useful remark is the equality $\pi_I' \circ P_{J/I} = P_J \circ \pi_I'$ on $(E/I)'$.

Suppose that $J \in F_X(E)$. Then for each $g \in (\pi_I(X))^\circ$ we have $\pi_I'(g) \in X^\circ$ and thus $(P_J \circ \pi_I')(g) \in X^\circ$ i.e., $(\pi_I' \circ P_{J/I})(g) \in X^\circ$. Consequently $P_{J/I}(g) \in \pi_I(X))^\circ$ i.e., $J/I$ is a $\pi_I(X)$-frontal ideal of $E/I$.

Conversely, suppose that that $J/I$ is a $\pi_I(X)$-frontal ideal and let $f \in X^\circ$. Since $I$ is a frontal ideal, $P_I(f) \in X^\circ$ and thus there exists a $g \in (E/I)'$ such that $P_I(f) = \pi_I'(g)$. Then $g \in (\pi_I(X))^\circ$ and $P_{J/I}(g) \in (\pi_I(X))^\circ$ because $J/I$ is a $\pi_I(X)$-frontal ideal. Consequently $(\pi_I' \circ P_{J/I})(g) \in X^\circ$ which yields

$$P_J(f) = (P_J \circ P_I)(f) = (P_J \circ \pi_I')(g) = (\pi_I' \circ P_{J/I})(g) \in X^\circ$$

i.e., $J \in F_X(E)$. ■

## 4. THE BISHOP-SILOV DECOMPOSITION

We start with a natural generalization of the notion of a set of anti-symmetry.

Let $E$ be a complex Banach space and let $X$ be a subspace of $E$.

**4.1** *Definition.* An $M$-ideal $I$ of $E$ is said to be *anti-symmetric* with respect to $X$, provided that every $U \in \operatorname{Re} Z(E/I)$ such that $U(\pi_I(X)) \subset \subset \pi_I(X)$ is a multiple of $1_{E/I}$.

For $E$ a space $C(K, \mathbb{C})$ and $X$ a function algebra on $K$ we retrieve the notion of a set of anti-symmetry.

We shall denote by $\mathscr{A}_X(E)$ the set of all $M$-ideals of $E$, anti-symmetric with respect to $X$. Clearly,

$$\mathscr{A}_X(E) \subset \mathscr{A}_{\overline{X}}(E).$$

The fact that every point belongs to a maximal set of anti-symmetry has the following analogue in terms of M-structure theory.

**4.2** LEMMA. *Suppose that $E$ is $M$-distinguished and let $(I_\alpha)_\alpha$ be a family of elements of $\mathscr{A}_X(E)$ such that $J = \overline{\mathrm{Span}} \cap I_\alpha \neq E$. Then $I = \cap\, I_\alpha \in \mathscr{A}_X(E)$.*

*Proof.* We start by noticing a canonical mapping relating to the centralizers.

Let $A$ and $B$ be two $M$-ideals of $E$ such that $B \subset A$. Then we can consider the mapping

$$M_{BA} : Z(E/B) \to Z(E/A)$$

given by

$$M_{BA}(U)(\pi_A(x)) = \pi_{BA}(U\pi_B(x))$$

for all $x \in A$, where $\pi_{BA}$ makes commutative the diagram

$$\begin{tikzcd} & E \arrow[dl, "\pi_B"'] \arrow[dr, "\pi_A"] & \\ E/A \arrow[rr, "\pi_{BA}"'] & & E/B \end{tikzcd}$$

Clearly, $M_{BA}$ maps $\operatorname{Re} Z(E/B)$ into $Z(E/A)$.

Now, let $U \in \operatorname{Re} Z(E/I)$ such that $U(\pi_I(X)) \subset \pi_I(X)$. Then for each $\alpha$, $M_{II_\alpha}(U)\,(\pi_{I_\alpha}(X)) = \pi_{II_\alpha}(U\pi_I(X)) \subset \pi_{II_\alpha}(\pi_I(X)) = \pi_{I_\alpha}(X)$. Since $I_\alpha \in \mathscr{A}_X(E)$, there exists an $a_\alpha \in \mathbb{R}$ such that $M_{II_\alpha}(U) = a_\alpha \cdot 1_{E/I_\alpha}$ and thus

$$M_{IJ}(U) = (M_{I_\alpha J} \circ M_{II_\alpha})(U) = a_\alpha \cdot 1_{E/J}.$$

As $E/J \neq 0$, it follows that $a_\alpha = a_\beta = a$ for all $\alpha$ and $\beta$. Then $M_{II_\alpha}(U - a \cdot 1_{E/I}) = 0$ for all $\alpha$ and thus $U = a \cdot 1_{E/I}$ i.e., $I \in \mathscr{A}_X(E)$.

**4.3** COROLLARY. *Suppose that $E$ is $M$-distinguished. Then every $I \in \mathscr{A}_X(E)$, $I \neq E$, contains a (unique) minimal ideal $I_0$ in $\mathscr{A}_X(E)$.*

*Proof.* In fact, $I_0 = \cap \{J \mid J \in \mathscr{A}_X(E),\ J \subset I\}$. ■

We shall show that the set $\tilde{\mathscr{A}}_X(E)$ of all minimal anti-symmetric ideals of $E$ can be viewed as an analogue of Bishop-Silov decomposition.

Assertion (i) in Theorem 1.3 constitutes the objective of Corollary 4.5 below.

**4.4** PROPOSITION. *Suppose that $E$ is $M$-distinguished and $X$ is a closed subspace of $E$. Then*

$$\tilde{\mathscr{A}}_X(E) \subset F_X(E).$$

*Proof.* Let $I \in \tilde{\mathscr{A}}_X(E)$ and $H = \overline{\mathrm{Span}} \cup \{J \mid J \in F_X(E),\ J \subset I\}$. By Lemma 2.2ii, $H \notin F_X(E)$. We shall show that $H = I$.

If $H \neq I$, then $H \in \mathscr{A}_X(E)$ i.e., there exists a $U \in \mathrm{Re}\ Z(E/H)$ such that $0 \leqslant U \leqslant 1_{E/H}$, $U \neq 1_{E/H}$, $\|U\| = 1$ and $U(\pi_H(X)) \subset \pi_H(X)$. We can also assume that $\|M(U)\| = 1$, where $M = M_{HI} : Z(E/H) \to Z(E/I)$ is the map considered in the proof of Lemma 4.2. This can be done by replacing $U$ (if necessary) by a polynomial $p(U)$ of $U$ where

$$0 \leqslant p(t) \leqslant 1 \quad \text{for } t \in [0,1],\ p(1) \neq 1 \text{ and } p(\|M(U)\|) = 1.$$

Put

$$L = \{y \mid y \in E/H,\ \lim_{n \to \infty} U^n y = 0\}.$$

$L$ is a closed subspace of $E/H$. In fact, let $x \in L$ and let $\mathscr{V}$ be an open neighbourhood of 0 in $E/H$. Then there exists an $x_1 \in L$ such that $x - x_1 \in \mathscr{V}$. Since $\|U\| \leqslant 1$, we infer that $U^n x - U^n x_1 \in \mathscr{V}$ for all $n \in \mathbb{N}$. Or, $U^n x_1 \to 0$, which yields $x \in L$.

$L \neq 0$. In fact, since $Z(E/H)$ can be though of as a $C(S)$-space, $\|U^n(1_{E/H} - U)\| \to 0$. Let $x \in E/H$ such that $Ux \neq x$. Then $\|U^n(1_{E/H} - U)x\| \to 0$ i.e., $x - Ux \in L \setminus \{0\}$.

$L = \overline{\mathrm{Im}}\ (I - U)$. In fact, the inclusion $\overline{\mathrm{Im}}\ (I - U)\ L$ is clear. For the other inclusion, let $x' \in (\overline{\mathrm{Im}}\ (I - U))^\circ = \mathrm{Ker}\ (I - U')$ and let $x \in L$. Then $\langle x', x\rangle = \langle U'^n x', x\rangle = \langle x', U^n x\rangle \to 0$ i.e., $x' \in L^\circ$.

By Proposition 2.4 we infer that $L$ is an $M$-ideal. The $L$-projection onto $L^\circ$ is $P = \lim_{n\to\infty} U'^n$. The fact that $P$ is an $L$-projection follows from [3], Proposition 3.11 ii. Clearly, $\mathrm{Im}\ P \subset L^\circ$. For the other inclusion, let $x' \in L^\circ$ and $x \in E$. Then

$$\langle x' - U'^n x', x\rangle = \langle x', (I - U^n)x\rangle \to 0$$

which shows that $x' = Px' \in \mathrm{Im}\ P$.

By Corollary 3.7, $\pi_H(X)$ is a closed subspace of $E/H$. Due to the form of $P$, it follows that $L$ is a $\pi_H(X)$-frontal ideal of $E/H$.

$L$ is contained in $I/H$. In fact, since $M(U)(\pi_I(X)) \subset \pi_I(X)$ and $I \in \mathscr{A}_X(E)$ it follows that $M(U) = a \cdot 1_{E/I}$ for a suitable $a \in \mathbb{R}$. Actually $a = 1$ because $M(U) \geqslant 0$ and $\|M(U)\| = 1$. Then $M(U)^n = 1_{E/I}$, which yields $U^n x - x \in I/H$ for every $x \in E/H$ and every $n \in \mathbb{N}$. Since $I/H$ is closed and $U^n x \to 0$ for $x \in L$, we conclude that $L \subset I/H$.

By Lemma 2.3 ii and the above remarks, $\pi_I^{-1}(L)$ is an $M$-ideal of $E$ such that $H \subsetneqq \pi_I^{-1}(L) \subset I$. By Proposition 3.9, $\pi_I^{-1}(L) \in F_X(E)$, in contradiction with the definition of $H$. Consequently $H = I$ and the proof is complete. ■

4.5 Corollary. *If $I \in \widetilde{\mathscr{A}}_X(E)$ then $\pi_I\ (X)$ is a closed subspace of $E/I$.*

The following result extends de Branges'Lemma:

4.6 Lemma. *Let $E$ be a Banach space, $X$ a proper subspace of $E$ and $f$ an extreme point of the unit ball of $X^\circ$. Then the maximal $M$-ideal $I$ contained in Ker $f$ is anti-symmetric with respect to $X$.*

*Proof.* Let $U \in Z(E/I)$ such that $0 \leqslant U \leqslant 1_{E/I}$ and $U(\pi_I(X)) \subset \subset \pi_I(X)$.

Since $I \subset \text{Ker } f$, there exists a unique $g \in (E/I)'$ such that $f = g \circ \pi_I$ and $\|g\| = 1$. Suppose that $U'(g) \neq 0$ and $U'(g) \neq g$. Then

$$g = \|U'(g)\| \cdot \frac{U'(g)}{\|U'(g)\|} + \|g - U'(g)\| \cdot \frac{g - U'(g)}{\|g - U'(g)\|}$$

which yields the following convex decomposition in $X^\circ$.

$$f = g \circ \pi_I = \|U'(g)\| \frac{(\pi_I' \circ U')(g)}{\|U'(g)\|} + \|g - U'(g)\| \frac{\pi_I'(1 - U')(g)}{\|g - U'g\|}.$$

Then $f = (\pi_I' \circ U')(g)/\|U'(g)\|$ i.e., $g = U'(g)/\|U'(g)\|$ and thus there exists a real scalar $\lambda$ such that $(\lambda 1_{(E/I)}' - U')(g) = 0$.

To end the proof we shall show that

$$(*) \qquad V \in Z(E/I) \text{ and } V'(g) = 0 \text{ implies } V = 0.$$

In fact, replacing $V$ by $V^2$ if necessary, we may assume in addition that $0 \leqslant V \leqslant 1_{E/I}$. By the definition of $I$, there exists no $M$-ideal $J \neq 0$ contained in Ker $g$. Or, $V'(g) = 0$ yields $\overline{\text{Im } V} \subset \text{Ker } g$ and $\overline{\text{Im } V}$ is an $M$-ideal. See Proposition 2.4 above. Consequently $V = 0$ and the assertion $(*)$ is proved.

By $(*)$, $U = \lambda \cdot 1_{E/I}$ for a suitable $\lambda \in \mathbb{R}$ and thus $I$ is anti-symmetric. ■

We can now state the main result of our paper:

4.7 Theorem. (Bishop-Silov decomposition for $M$-distinguished spaces). *Suppose that $E$ is $M$-distinguished and $X$ is a closed subspace of $E$. Then*

(i) *$\pi_I(X)$ is a closed subspace of $E/I$ for every $I \in \widetilde{\mathscr{A}}_X(E)$;*

(ii) *For every $x \in E$,*

$$d(x, X) = \sup \{d(\pi_I(x), \pi_I(X)) \mid I \in \widetilde{\mathscr{A}}_X(E)\}.$$

*Proof.* The assertion (i) follows from Corollary 4.5 above.

(ii) Clearly, the non-trivial case is $X \neq E$. We shall denote by $K$ the closed unit ball of $E'$ and by Ext $K$ the subset of all extreme points of $K$. By Hahn-Banach Extension Theorem and Krein-Milman Theorem,

$$d(x, X) = \sup \{|x'(x)| \mid x' \in K \cap X^\circ\} =$$

$$= \sup \{|x'(x))| \mid x' \in \operatorname{Ext} (K \cap X^\circ)\}.$$

Let $x' \in \operatorname{Ext}(K \cap X^\circ)$. By Lemma 4.6, the maximal $M$-ideal $I$ contained in $\operatorname{Ker} x'$ belongs to $\mathscr{A}_X(E)$. By Corollary 4.3, $I$ contains an ideal $J \in \tilde{\mathscr{A}}_X(E)$. Since $J \subset \operatorname{Ker} x'$, there exists a unique functional $z' \in (E/J)'$ such that $z' \circ \pi_J = x'$. Then

$$|x'(x)| = |z'(\pi_J(x))| \leqslant d(\pi_J(x), \quad \pi_J(X)).$$

The other inequality,

$$\sup d(\pi_I(x), \ \pi_I(X)) \leqslant d(x, X),$$

is obvious. ■

It is important to comment the above results in the framework of $C^*$-algebras.

Let $\mathscr{U}$ be a $C^*$-algebra and let Prim $\mathscr{U}$ be the set of all primitive closed two sided ideals of $\mathscr{U}$. Prim $\mathscr{U}$ can be endowed with the Jacobson topology, consisting of all complements of hulls (a hull being the set of all primitive closed two-sided ideals containing some fixed closed two sided ideal). Let $C_b(\operatorname{Prim} \mathscr{U}, \mathbb{R})$ be the Banach space of all bounded continuous functions $f : \operatorname{Prim} \mathscr{U} \to \mathbb{R}$ endowed with the sup norm. By Lemma 4.6,

$$\operatorname{Prim} \mathscr{U} \subset \mathscr{A}_0(\mathscr{U})$$

which yields a natural one-to-one mapping

$$\varphi : \operatorname{Re} Z(\mathscr{U}) \to C_b(\operatorname{Prim} \mathscr{U}, \ \mathbb{R})$$

given by $\Phi(U)(I) = a$ provided $M_{0,I}(U) = a \cdot 1_{E/I}$. Actually $\Phi$ is an isomorphism. This fact is known as the Dauns-Hofmann Theorem. See [7] for details.

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